Single-Tone Parameter Estimation from Discrete-Time Observations

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Abstract—**Estimation of the parameters of a single-frequency complex tone from a finite number of noisy discrete-time observations is discussed. The appropriate Cramér–Rao bounds and maximum-likelihood (ML) estimation algorithms are derived. Some properties of the ML estimators are proved. *The relationship of ML estimation to the discrete Fourier* transform is exploited to obtain practical algorithms. The threshold effect of one algorithm is analyzed and compared to simulation results. Other simulation results verify other aspects of the analysis.**

I. Introduction

THIS PAPER discusses the problem of estimating the parameters of single-frequency tones from a finite number of noisy discrete-time observations. The problem has application to data set testing, telephone transmission system testing, radar, and other measurement situations.

The parameter estimation problem was formulated by Slepian [1]. His paper and most subsequent works have concentrated on the continuous-time observation model. The cases of discrete-time observations, particularly the one studied here, have received less attention.

In general the signal has the form $\sum_{i=1}^{k} b_i \exp [j(\omega_i t + \theta_i)]$. In a working system, the imaginary part may be derived from the real part by a Hilbert transformation or perhaps not be processed at all. We assume the signal and noise are band limited.

In this discussion, we will concentrate on the case of a single tone, in which real and imaginary parts are both processed; that is, $k = 1$. An understanding of this case is fundamental to an understanding of the general case. We have studied the general case of many tones in addition to the case presented here [2], and plan to present it in another paper.

The real part of the signal $s(t)$ is $b_0 \cos (\omega_0 t + \theta_0)$. Suppose some or all of the parameters are unknown. The computer input will be two sample vectors: $X = [X_0, X_1, \cdots, X_{N-1}]^T$ and $Y = [Y_0, Y_1, \cdots, Y_{N-1}]^T$, where

$$X_n = s(t_n) + W(t_n), \qquad 0 \le n \le N-1 \tag{1}$$

$$Y_n = \check{s}(t_n) + \check{W}(t_n), \qquad 0 \le n \le N-1 \tag{2}$$

and

$$\check{s}(t) = b_0 \sin (\omega_0 t + \theta_0). \tag{3}$$

We assume a constant sampling rate of $1/T$ with the first sample taken at $t = t_0$. Thus

$$t_n = t_0 + nT = (n_0 + n)T. \tag{4}$$

$\check{W}(t)$ is the Hilbert transform of the noise $W(t)$. We consider only the case of independent Gaussian noise samples with zero mean and variance σ^2.

If we write $Z = X + jY$, then the joint probability density function (pdf) of the elements of the sample vector Z when the unknown parameter vector is α is given by

$$f(Z; \alpha) = \left(\frac{1}{\sigma^2 2\pi}\right)^N \exp \left[-\frac{1}{2\sigma^2} \sum_{n=0}^{N-1} (X_n - \mu_n)^2 + (Y_n - \nu_n)^2 \right] \tag{5}$$

where, if ω, b, and θ are all unknown,

$$\alpha = [\omega, b, \theta]^T \tag{6}$$

$$\mu_n = b \cos (\omega t_n + \theta) \tag{7}$$

$$\nu_n = b \sin (\omega t_n + \theta). \tag{8}$$

In developing the topic we will consider three main aspects of the problem. First, we examine Cramér–Rao (CR) lower bounds to estimation error. Then we develop and analyze maximum-likelihood (ML) estimators of the signal parameters. Finally, we discuss practical estimation algorithms and simulation results. The frequency estimation algorithm has a threshold effect, which we also discuss.

Palmer used the same model in [3]. However, his approach was different and he obtained different results. The paper by Gumacos is also related [4].

II. Bounds

In an estimation (or measurement) system, it is important to have numbers that indicate the best estimation that can be made with the available data (the observations). The rms errors are important and are often used as a measure of system inaccuracy. Estimation bias is of secondary importance, although it is generally desirable to minimize bias. In this paper, we will find that for our purposes the bias can usually be neglected. Thus rms errors will be the important consideration. We will use ML estimation and will generally be able to keep the bias very small. Thus, above threshold, the unbiased CR bound will apply. We will separately evaluate threshold effects.

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The unbiased CR bounds are the diagonal elements of the inverse of the Fisher information matrix J, whose typical element is given by

$$J_{ij} = E\{H_{\alpha_i}H_{\alpha_j}\} = -E\{H_{\alpha_i\alpha_j}\} \tag{9}$$

where the expectation is with respect to the sample vector $\boldsymbol{Z}$ and

$$H_{\alpha_i} = \frac{\partial}{\partial \alpha_i} \log f(\boldsymbol{Z}; \boldsymbol{\alpha}). \tag{10}$$

The bounds are given by

$$\mathrm{var}\,\{\hat{\alpha}_i\} \geq J^{ii}, \tag{11}$$

where $\hat{\alpha}_i$ is the estimator of α_i and J^{ii} is the ith diagonal element of J^{-1}.

When $f(\boldsymbol{Z}; \boldsymbol{\alpha})$ is given by (5), the elements of J are

$$J_{ij} = \frac{1}{\sigma^2} \sum_{n=0}^{N-1} \left[\frac{\partial \mu_n}{\partial \alpha_i}\frac{\partial \mu_n}{\partial \alpha_j} + \frac{\partial \nu_n}{\partial \alpha_i}\frac{\partial \nu_n}{\partial \alpha_j}\right]. \tag{12}$$

The subscripts i and j in (12) should refer to only the unknown elements in $\boldsymbol{\alpha}$. For example, if two of the three elements in $\boldsymbol{\alpha}$ are unknown, then J is a 2-by-2 matrix.

The most general case is of all elements of $\boldsymbol{\alpha}$ unknown. The matrix J, from (12), is then

$$J = \frac{1}{\sigma^2}\begin{bmatrix} b_0^2T^2(n_0^2N + 2n_0P + Q) & 0 & b_0^2T(n_0N + P) \\ 0 & N & 0 \\ b_0^2T(n_0N + P) & 0 & b_0^2N \end{bmatrix} \tag{13}$$

where

$$P = \sum_{n=0}^{N-1} n = \frac{N(N-1)}{2} \tag{14}$$

$$Q = \sum_{n=0}^{N-1} n^2 = \frac{N(N-1)(2N-1)}{6} \tag{15}$$

and $t_0 = n_0T$ is the time at which the first sample is taken.

J can be obtained from (13) for all combinations of unknown parameters. If the phase is known, for example, then J is the 2-by-2 matrix obtained by deleting the third row and third column from (13).

After inverting all the variations of J, corresponding to different unknown parameters, one obtains the following set of bounds:

$$\mathrm{var}\,\{\hat{\omega}\} \geq \begin{cases} \dfrac{\sigma^2}{b_0^2T^2(n_0^2N + 2n_0P + Q)}, & \text{phase is known and amplitude known or not} \quad (16) \\[2ex] \dfrac{12\sigma^2}{b_0^2T^2N(N^2 - 1)}, & \text{phase is unknown and amplitude known or not} \quad (17) \end{cases}$$

$$\mathrm{var}\,\{\hat{b}\} \geq \frac{\sigma^2}{N}, \qquad \text{in all cases} \tag{18}$$

and

$$\mathrm{var}\,\{\hat{\theta}\} \geq \begin{cases} \dfrac{\sigma^2}{b_0^2N}, & \text{frequency is known and amplitude known or not} \quad (19) \\[2ex] \dfrac{12\sigma^2(n_0^2N + 2n_0P + Q)}{b_0^2N^2(N^2 - 1)}, & \text{frequency is unknown and amplitude known or not.} \quad (20) \end{cases}$$

Line (17) is equivalent to a result obtained by Brennan [5, eq. (14)] in connection with angular measurement accuracy of a phased-array radar.

We see that if the phase is known, then the frequency bound depends upon n_0. It is easy to show that if the sampling times are symmetrically located about zero, i.e.,

$$t_0 = -\left(\frac{N-1}{2}\right)T \tag{21}$$

then the frequency bound attains its maximum value. This maximum is the same as the bound when the phase is unknown. On the other hand, the further in time between the instant at which the angle is known (where $t = 0$) and when the samples are taken, the more accurately the frequency can be estimated. Simulation results, discussed in Section V, verify this result. In most problems, we do not expect to know the phase and cannot take advantage of the preceding property of frequency estimates.

If the frequency is known, the phase bound is independent of t_0. If the frequency is not known then the phase bound depends upon t_0. The minimum bound is obtained if the t_0 is given by (21) and equals the bound when the frequency is known.

The dependence of bounds upon the time at which the first sample is taken is inherent in discussions presented in the well-known texts on the subject, but is generally not mentioned. See, for example, Van Trees, [6, pp. 273–286] where the subject is not discussed; and Seidman, [7, pp. 91 and 92] where it is. Seidman indicates that threshold effects are also a function of t_0 when the signal phase is known.

The CR bounds are almost met by ML estimators when the SNR is "high." Thus all of the properties given by (16)–(20) can be verified by simulations.

III. Maximum-Likelihood Estimation

Now let us turn to ML estimation. We will discuss in detail the case when all three parameters are unknown. The results for the other cases will be stated without proof.

A. General

The ML estimate of $\boldsymbol{\alpha}$ is the value of $\boldsymbol{\alpha}$, say $\hat{\boldsymbol{\alpha}}$, that maximizes $f(\boldsymbol{Z}; \boldsymbol{\alpha})$ when $\boldsymbol{Z}$ is the observed sample vector. The maximum of $f(\boldsymbol{Z}; \boldsymbol{\alpha})$ will occur at the maximum of $\log(f)$, or, using (5), at the maximum of

$$L_0 = -\frac{1}{N} \sum_{n=1}^{N-1} (X_n - \mu_n)^2 + (Y_n - \nu_n)^2 \tag{22}$$

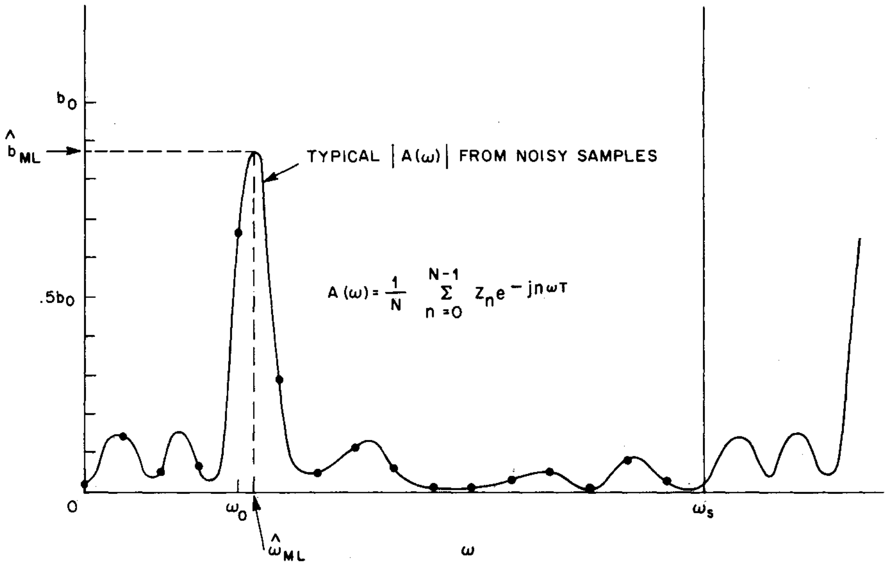

Fig. 1. Maximum-likelihood estimation.

where μ_n and ν_n were defined earlier. Since $\sum X_n^2$ and $\sum Y_n^2$ are constants once an observation has been made, we can drop them from L_0 and maximize L:

$$L = \frac{2}{N} \sum_n (X_n \mu_n + Y_n \nu_n) - \frac{1}{N} \sum_n (\mu_n^2 + \nu_n^2). \tag{23}$$

After substitution of the definitions of μ_n and ν_n into (23), and some rearrangement, we get

$$L = 2b \operatorname{Re} [\exp(-j\theta) \exp(-j\omega t_0) A(\omega)] - b^2 \tag{24}$$

where

$$A(\omega) = \frac{1}{N} \sum_{n=0}^{N-1} Z_n \exp(-jn\omega T) \tag{25}$$

and Re $[\cdot]$ means real part of $[\cdot]$.

B. All Parameters Unknown

Now suppose all three of the parameters are unknown and $b_0 > 0$.

It is easy to show that L is maximized over θ, for a fixed ω, if $\theta = \arg [\exp(-j\omega t_0) A(\omega)]$, where arg $[\cdot]$ means the argument or phase of $[\cdot]$, taken mod 2π for convenience. Then we obtain

$$\max_{\theta} L = 2b|A(\omega)| - b^2. \tag{26}$$

Let $\hat{\omega}$ be the value of ω that maximizes $|A(\omega)|$. Then assuming $b > 0$,

$$\max_{\theta,\omega} L = 2b|A(\hat{\omega})| - b^2. \tag{27}$$

Finally, the value of b that maximizes (27) is

$$\hat{b} = |A(\hat{\omega})| \tag{28}$$

which gives

$$\max_{\theta,\omega,b} L = |A(\hat{\omega})|^2. \tag{29}$$

The numbers $\hat{\omega}$ and $\hat{b}$ are the ML estimates of ω_0 and b_0.

Returning to θ, the ML estimate of θ is

$$\hat{\theta} = \arg [\exp(-j\hat{\omega} t_0) A(\hat{\omega})]. \tag{30}$$

Observe that $\hat{\omega}$ and $\hat{b}$ do not depend explicitly on t_0, but $\hat{\theta}$ does. This is to be expected because of the way the CR bounds depend upon t_0.

The $\hat{b}$ and $\hat{\omega}$ algorithms for this case (all parameters unknown) are illustrated in Fig. 1. On the figure, N is 16.

The function $A(\omega)$ is periodic in ω with period $\omega_s = 2\pi/T$. Thus the $\hat{\omega}$ algorithm must be used mod ω_s. Normally the input signal would be passed through a low-pass filter to assure that all input frequencies are less than ω_s.

Relationship to Discrete Fourier Transform: Recall that the discrete Fourier transform (DFT) of the vector $\boldsymbol{Z}$ is the set of complex numbers

$$A_k = \frac{1}{N} \sum_{n=0}^{N-1} Z_n \exp\left(-\frac{j2\pi nk}{N}\right), \quad k = 0,1,2,\cdots, N-1. \tag{31}$$

From (31) and the definition of $A(\omega)$,

$$A_k = A\left(\frac{2\pi k}{NT}\right), \quad k = 0,1,\cdots, N-1. \tag{32}$$

The dots along the curve on Fig. 1 are the $\{|A_k|\}$ points. This relationship suggests that coarse (approximate) estimates of $\hat{\omega}_{ML}$ and $\hat{b}_{ML}$ can be made directly from the DFT of $\boldsymbol{Z}$ as was done by Palmer [3]. A fast Fourier transform (FFT) renders the calculation of the set $\{A_k\}$ fairly rapidly.

The reader is referred to Bergland [8], Cochran, *et al.* [9], and Cooley *et al.* [10], [11] for discussions of the DFT and FFT, and to Rife and Vincent [12] for means of extracting frequency and level estimates from the DFT.

C. Summary of Algorithms

The ML algorithms for all combinations of unknown parameters can be derived in the manner just described. The results will be summarized.

If ω_0 is unknown then $\hat{\omega}_{ML}$ maximizes

1) $\mathrm{Re}\,[\exp(-j\theta_0)\exp(-j\omega t_0)A(\omega)]$, if phase is known,
2) $|A(\omega)|$, if phase is unknown.

If b_0 is unknown then $\hat{b}_{ML}$ is equal to

3) $\mathrm{Re}\,[\exp(-j\theta_0)\exp(-j\omega_0 t_0)A(\omega_0)]$, if frequency and phase are known,
4) $\mathrm{Re}\,[\exp(-j\theta_0)\exp(-j\hat{\omega}t_0)A(\hat{\omega})]$, if phase is known but frequency is unknown
5) $|A(\omega_0)|$, if phase is unknown but frequency is known, or
6) $|A(\hat{\omega})|$, if frequency and phase are unknown.

One can show that $\hat{b}$ given by 3) is normally distributed with mean b_0 and variance equal to the CR bound σ^2/N.

Finally, $\hat{\theta}_{ML}$ is equal to

7) $\arg\,[\exp(-j\omega_0 t_0)A(\omega_0)]$, if frequency is known, or
8) $\arg\,[\exp(-j\hat{\omega}t_0)A(\hat{\omega})]$, if frequency is unknown.

In 4), $\hat{\omega}$ is from 1). In 6) and 8), $\hat{\omega}$ is from 2).

D. *Properties of* $\hat{\omega}$

The ML estimates of ω_0 have the following properties.

1) The pdf of $\hat{\omega}$ is symmetrical about ω_0, mod ω_s.
2) $\mathrm{var}\,\{\hat{\omega}\}$ is proportional to ω_s^2 and independent of θ_0.

We will prove these statements for the phase-unknown case. The proof for the phase-known case is similar. When the phase is unknown $\mathrm{var}\,\{\hat{\omega}\}$ is also independent of t_0, just as its CR bound is.

Noise Model: The following noise model is convenient. Let $\{V_n\}$ be a set of independent Rayleigh random variables with parameter 1. That is

$$f_{V_n}(v) = \begin{cases} v\exp[-v^2/2], & v \geq 0 \\ 0, & v < 0. \end{cases} \tag{33}$$

Let $\{\phi_n\}$ be a set of independent random variable uniformly distributed over $(-\pi, \pi)$.

We model the Gaussian samples as

$$W_n = \sigma V_n \cos\phi_n \tag{34}$$

and

$$\check{W}_n = \sigma V_n \sin\phi_n. \tag{35}$$

Proof: Recall that $Z_n = X_n + jY_n$. Then, using the noise model,

$$Z_n = b_0 \exp[j(n\omega_0 T + \omega_0 t_0 + \theta_0)] + \sigma V_n \exp(j\phi_n). \tag{36}$$

Thus

$$A(\omega) = \frac{1}{N}\exp[j(\theta_0 + \omega_0 t_0)] \cdot \sum_n [b_0 \exp(-jn\beta) + \sigma V_n \exp[-j(n\beta - \gamma_n)]] \tag{37}$$

where

$$\beta = (\omega - \omega_0)T \tag{38}$$

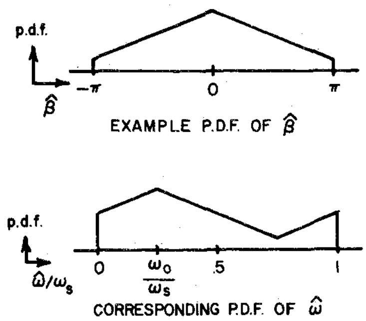

Fig. 2. Relationship of pdf of $\hat{\beta}$ to pdf of $\hat{\omega}$.

and

$$\gamma_n = \phi_n - \theta_0 - \omega_0 t_0 - n\omega_0 T. \tag{39}$$

Since the ϕ_n are independent and uniformly distributed on $(-\pi,\pi)$, in effect so are the γ_n.

From (37), $|A(\omega)|$ is not a function of θ_0 or t_0.

Without loss of generality, let $\hat{\beta}$ be the value of β in the range $(-\pi,\pi)$ that maximizes $|A(\omega)|$. The ML estimate $\hat{\omega}$, will then have the value:

$$\hat{\omega} = \omega_0 + \frac{\omega_s \hat{\beta}}{2\pi}, \qquad \mathrm{mod}\ \omega_s. \tag{40}$$

Observe that $|A(\omega)|$ is an even function of the pair (β,γ). The statistics of $-\gamma$ are the same as the statistics of γ. Thus the statistics of $-\hat{\beta}$ must be the same as the statistics of $\hat{\beta}$. Hence the pdf of $\hat{\beta}$ must be an even function of $\hat{\beta}$ and $E\{\hat{\beta}\} = 0$. From (37), the statistics of $\hat{\beta}$ do not depend upon ω_0 or θ_0, but do depend upon the SNR $b_0/2\sigma^2$.

Discussion: Since we choose $\hat{\omega}$ according to (40), the pdf of $\hat{\beta}$ is related to the pdf of $\hat{\omega}$ in the manner illustrated in Fig. 2. The pdf of $\hat{\omega}$ is even about ω_0 except for the part from $2\omega_0$ to ω_s, when $\omega_0 < \omega_s/2$ (or the part from 0 to $2\omega_0 - \omega_s$, when $\omega_0 > \omega_s/2$).

Consider the situation when $\omega_0 < \omega_s/2$. If $\Pr\{2\omega_0 < \hat{\omega} < \omega\}$ is small, which it is when the SNR is large enough, then $E\{\hat{\omega}\} \approx \omega_0$ or $\hat{\omega}$ is unbiased. If $\Pr\{2\omega_0 < \hat{\omega} < \omega_s\}$ is significant then $\hat{\omega}$ is biased in the direction of $\omega_s/2$. In other words, $E\{\hat{\omega} - \omega_0\} > 0$. If $\omega_0 > \omega_s/2$ the preceding remarks reply with the obvious modifications. Observe that due to the symmetry of the problem, the bias of $\hat{\omega}$ must be an odd function of ω_0, about $\omega_s/2$.

It is easy to show that if ω_0 is equal to zero, $\omega_s/2$, or ω_s, the pdf of $\hat{\omega}$ is even about $\omega_s/2$. Thus in these three cases $E(\hat{\omega}) = \omega_s/2$. We see, therefore, that the bias of $\hat{\omega}$ has the following values

ω_0	$E(\hat{\omega} - \omega_0)$
0	$\omega_s/2$
$\omega_s/2$	0
ω_s	$-\omega_s/2$

Clearly we expect to make large frequency estimation errors if ω_0 is close to zero or ω_s. At moderate SNR, say above the threshold region, we found that large errors did not occur if the difference between ω_0 and zero (or ω_s) was at least four times the rms CR bound.

The variance of $\hat{\beta}$ depends only upon the SNR. Thus the variance of $\hat{\omega}$ is proportional to ω_s^2 and is not a function of θ_0. The variance of $\hat{\omega}$ is a function of ω_0, but its variation with ω_0 is small at SNR above the threshold region. Hence the CR bound for unbiased estimators is appropriate in this region.

IV. Algorithm

We want to discuss threshold effects and simulation results. Before we do so, however, it is necessary to dwell upon some practical estimation algorithm details.

As indicated, once an estimate of ω_0 is made, estimates of b_0 and θ_0 can be done by straightforward computations, using appropriate equations. Thus the difficult and time-consuming part of an algorithm is the part that locates the maximum of $|A(\omega)|$. This part is essentially a search routine.

One way to develop an algorithm is to use a two-part search routine. The first part calculates $|A(\omega)|$ for a set of ω values between zero and ω_s, and identifies the ω that maximizes $|A(\omega)|$ over this set of ω values. The second part locates the local maximum closest to the value of ω picked out by the first part. We call the first part the *coarse search* and the second part the *fine search*. If the coarse search is organized properly, this procedure will almost always locate the global maximum of $|A(\omega)|$ and thus the ML estimates.

When there is no noise it can be shown that

$$A(\omega) = b_0 \exp\left[j(\theta_0 + \omega_0 t_0)\right] \exp\left[-j(N-1)z\right] \frac{\sin Nz}{N \sin z} \tag{41}$$

where

$$z = \frac{\omega - \omega_0}{2} T = \frac{\pi(\omega - \omega_0)}{\omega_s}. \tag{42}$$

Thus

$$|A(\omega)| = b_0 \left| \frac{\sin(Nz)}{N \sin(z)} \right|. \tag{43}$$

This function is symmetric about ω_0 and has period ω_s. The global maximum occurs at ω_0 and has value b_0. There are also numerous low amplitude maxima. Without noise the ML estimates of ω_0 and b_0 have no error.

When noise is present $|A(\omega)|$ loses its clean, symmetrical shape and the minor maxima get larger. The global maximum is usually close to ω_0.

If the SNR is small, $|A(\omega)|$ will occasionally be so badly distorted that the global maximum occurs at a frequency far removed from ω_0. When this happens, the ML frequency estimation algorithm makes a large error. It is the occurrence of these rare but large errors, which we call *outliers*, at low SNR that causes var $\{\hat{\omega}_0\}$ to be much larger than the CR bound.

The Coarse Search: For our coarse search we evaluate $|A(\omega)|$ at the set of frequencies $\{\omega_k\}$ defined by

$$\omega_k = \frac{2\pi k}{MT}, \qquad k = 0,1,2,\cdots, M-1 \tag{44}$$

when M is $2N$, $4N$, or $8N$. We always choose N to be a power of 2.

Observe that the set $\{A(\omega_k)\}$ is the DFT of the set $\{\bar{Z}_n\}$ defined by

$$\bar{Z}_n = \begin{cases} (M/N)Z_n, & n = 0,1,2,\cdots, N-1 \\ 0, & n = N, N+1,\cdots, M-1. \end{cases} \tag{45}$$

The output of the coarse search is the value of ω_k, say ω_l, that corresponds to the largest member of the set $\{|A(\omega_k)|\}$.

It would seem natural to use $M = N$ in the coarse search; that is, to use the DFT of the observed data. However, it turned out that the ω_l thus obtained was the wrong choice (not close to the global maximum) often enough at low SNR to cause trouble. We found that the number of wrong choices was significantly reduced when the coarse search used M/N equal to 2 or 4.

The Fine Search: The fine search algorithm locates the value of ω closest to ω_l that maximizes $|A(\omega)|$. We used the *secant method* [13, p. 52] to compute successive approximations to the frequency estimate $\hat{\omega}$. The details are described in [2].

V. Threshold Effect and Simulations

A. General

It is well known that nonlinear estimation is generally plagued by threshold effects. At low SNR, there is usually a range of SNR in which the mean-squared error (mse) rises very rapidly as SNR decreases. The SNR at which this effect is first apparent is called the threshold. Receivers are often said to operate above or below threshold.

Digital frequency estimation also has threshold effects, connected with the occurrence of outliers. In this section we present a calculation of threshold effects. The result accurately describes one particular model.

Consider the estimation of the frequency of a single complex tone. Assume the phase is unknown. Suppose the tone frequency is $\omega_0 = \omega_s/2$. Assume the algorithm is the one described previously, with $M = N$.

Since $\omega_0 = \omega_s/2$, $|A_{N/2}|$ should be the largest. That is, the coarse search should give $l = N/2$. If $l \neq N/2$ we say an outlier has occurred.

We will approximate the mse when $l = N/2$ by CR bound for an unbiased estimator, which we designate ω_{CR}^2. From (17),

$$\omega_{\mathrm{CR}}^2 = \frac{3\omega_s^2}{2\pi^2 \rho N(N^2-1)} \tag{46}$$

where

$$\rho = b_0^2/2\sigma^2. \tag{47}$$

If an outlier occurs, the outcome of the fine search will be any frequency between zero and ω_s. The pdf is approximately uniform because the signal has little influence. Thus we write the mse when an outlier occurs as

$$\omega_{\mathrm{out}}^2 = \frac{\omega_s^2}{12}. \tag{48}$$

The total mse is the weighted sum of the two contributions,

$$\text{mse} = (\text{mse/outlier})q + (\text{mse/no outlier})(1-q) \tag{49}$$

where q is the probability of an outlier. Let the total mse be ω_e^2. Then we have

$$\omega_e^2 \approx q \frac{\omega_s^2}{12} + (1 - q) \frac{3\omega_s^2}{2\rho\pi^2 N(N^2 - 1)}. \quad (50)$$

The rms error is

$$\omega_{\text{rms}} = \sqrt{\omega_e^2}. \quad (51)$$

Next we calculate the probability of an outlier q and verify that when an outlier occurs, all possible l except the correct one are equally likely.

B. Probability of an Outlier

Let

$$C_k = |A_k|, \qquad k = 0 \text{ to } N - 1 \quad (52)$$

where A_k was defined. When both signal and noise are present, each C_k is a random variable. If the signal frequency is $\omega_s/2$ and the noise samples are independent, normal, and zero mean with variance σ^2 then it can be shown that the C_k are independent with Rayleigh distribution when $k \neq N/2$ and Rician distribution when $k = r = N/2$. Thus we have

$$f_k(C_k) = \frac{NC_k}{\sigma^2} \exp\left(-\frac{NC_k^2}{2\sigma^2}\right), \qquad C_k \geq 0, \quad k \neq \frac{N}{2} \quad (53)$$

and

$$f_r(C_r) = \frac{NC_r}{\sigma^2} \exp\left[-\frac{N(C_r^2 + b_0^2)}{2\sigma^2}\right] I_0\left(\frac{Nb_0C_r}{\sigma^2}\right), \qquad C_r \geq 0 \quad (54)$$

where $I_0(\cdot)$ is the modified Bessel function of the first kind. Then

$$1 - q = \Pr\{\text{all } C_k < C_r\} = \int_x \Pr\{\text{all } C_k < C_r \mid C_r = x\} \Pr\{C_r = x\}\, dx. \quad (55)$$

But

$$\Pr\{\text{all } C_k < C_r \mid C_r = x\} = [\Pr\{C_1 < C_r \mid C_r = x\}]^{N-1}. \quad (56)$$

Thus

$$1 - q = \int_0^\infty f_r(x) \left[\int_0^x f_k(y)\, dy\right]^{N-1} dx. \quad (57)$$

But

$$\int_0^x f_k(y)\, dy = \int_0^x \frac{Ny}{\sigma^2} \exp\left(-\frac{Ny^2}{2\sigma^2}\right) dy = 1 - \exp\left(-\frac{Nx^2}{2\sigma^2}\right). \quad (58)$$

Thus

$$1 - q = \int_0^\infty \left[1 - \exp\left(-\frac{Nx^2}{2\sigma^2}\right)\right]^{N-1} \cdot \frac{Nx}{\sigma^2} \exp\left[-\frac{N(x^2 + b_0^2)}{2\sigma^2}\right] I_0\left(\frac{b_0 x}{\sigma^2}\right) dx. \quad (59)$$

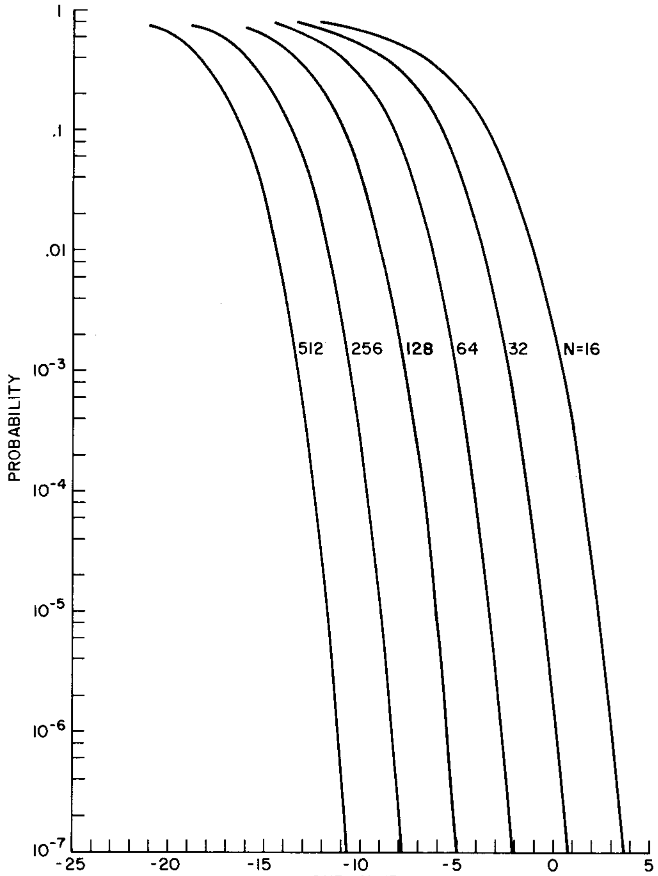

Fig. 3. Probability of outlier versus SNR.

After some further work, we obtain

$$q = \frac{1}{N} \sum_{m=2}^{N} \frac{N!\,(-1)^m}{(N - m)!\, m!} \exp\left(-N\rho \frac{m - 1}{m}\right). \quad (60)$$

The given formula for q cannot be easily summed because the terms $N!(-1)^m/n - m)!m!$ get very large and alternate in sign. Thus to compute q it was necessary to use the integral form and do numerical integration. The calculated values of q are shown in Fig. 3.

C. Approximate RMS Frequency Error

We used the preceding formula for ω_{rms} to compute the rms error for several values of N as shown in Fig. 4. The small circles on the curves represent the results of simulations. As can be seen, the simulation results agree with the calculated curves. The curves are similar to the well-known results for the continuous observation case (see Van Trees [6, p. 285]).

One would not usually operate a system at SNR below the threshold. Thus Fig. 4 is useful mainly because it shows the SNR at which the threshold effect starts. All SNR above threshold can be considered to be "high SNR" in the sense that the variance of ML estimators equals the CR bounds at high SNR.

The simulations described in the preceding included level estimates according to 6) (Section III-C). In every case the rms level errors were almost equal to the CR bounds. Threshold effects were not observed.

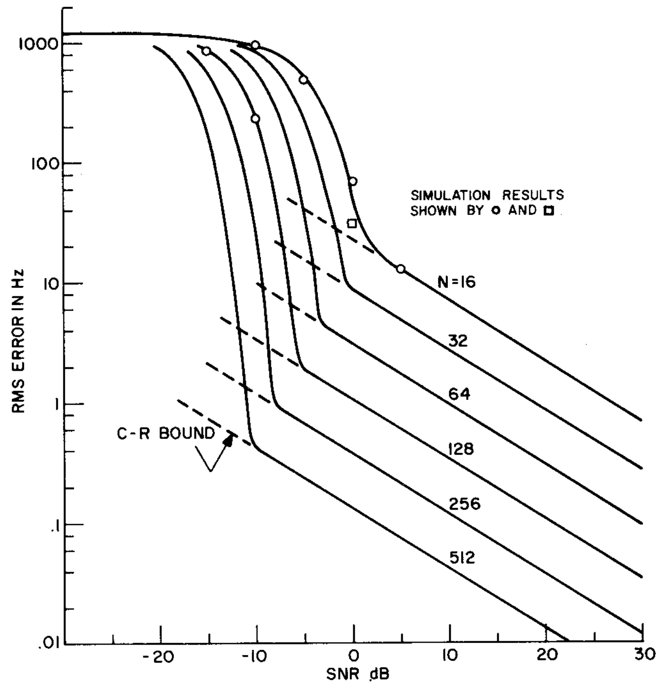

Fig. 4. Approximate performance of ML frequency estimate of single complex tone at 2000 Hz, with unknown phase. $1/T$ is 4000 Hz.

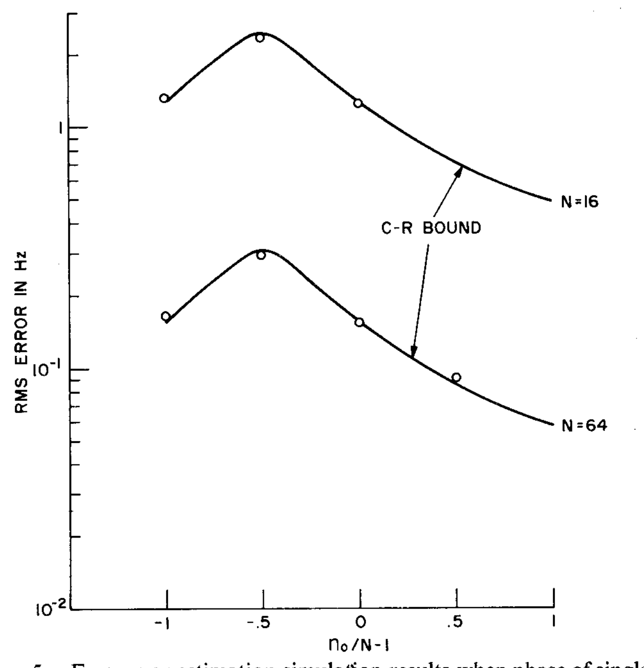

Fig. 5. Frequency estimation simulation results when phase of single complex tone is known. SNR is 20 dB. $1/T$ is 4000 Hz.

We ran the preceding simulations with $M = 4N$ instead of $M = N$. There was no significant difference in the results. Since using $M = 4N$ is more likely to result in correctly locating the global maximum in $|A(\omega)|$, we are led to believe that Fig. 4 truly depicts ML estimation when the frequency is one half the sampling frequency.

The next question is, what about different signal frequencies? We ran the simulation with $M = 64$, $N = 16$, and $f_0 = 2120$ Hz, using -10, -5, 0, and 5-dB SNR. The only point different from the ○ points is the □ point in Fig. 4. As before, level estimates did not show a threshold effect.

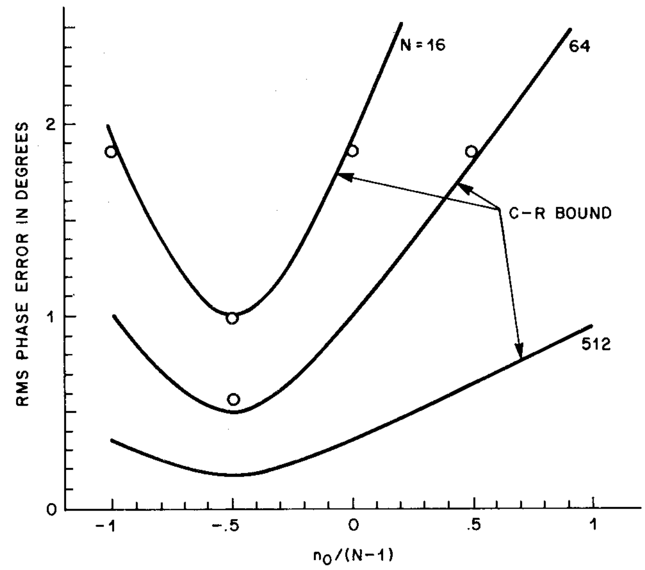

Fig. 6. Phase estimation simulation results when frequency is unknown. SNR is 20 dB. $1/T$ is 4000 Hz.

D. *Effect of* t_0

Line (16) tells us that when the phase is known, the variances of ML frequency estimations at high SNR will depend upon the time at which the first sample is taken t_0. We have simulated the $\hat{\omega}$ algorithm given by 1) (Section III-C). The results, shown in Fig. 5, verify the dependence on t_0.

Equation (20) shows that the variances of ML phase estimates will depend upon t_0. This, too, has been simulated, using the algorithm given by (28) and by 8) (Section III-C). The simulation results, shown in Fig. 6, verify the dependence upon t_0.

In a communication environment the phase of an arriving signal is almost never known. Thus the possibility of making arbitrarily good frequency estimates, by using large t_0 (and a perfect clock), is not realizable.

On the other hand, it is very common for both the frequency and phase of a received signal to be unknown. Thus adjustment of t_0 to minimize the phase estimation error is practical. The correct choice for t_0, as was mentioned above, is $-[(N - 1)/2]T$. This optimum value of t_0 offers roughly a four-to-one reduction in the variance of the phase estimator over the natural choice of $t_0 = 0$.

VI. Summary

This has been an introductory study of the problem of estimating the frequency and level of a cysoidal (complex sinusoidal) signal from a finite number of noisy observations of the signal. We derived the equations that describe the CR lower bounds to the variance of estimation errors. Then we derived the ML estimators and showed their relationship to the DFT. The analysis of the ML estimators revealed some of their properties. We presented an algorithm suitable for implementation on a digital computer. The algorithm almost always yields ML estimates. We were able to derive an expression for the threshold behavior of the algorithm. Simulation results verified the analysis.

The results of this paper justify and support the current use of the DFT for tone parameter estimation. We see, for example, that the interpolation algorithms proposed by Rife and Vincent [12] yield estimates that are close to ML when the noise samples are independent and Gaussian. See also Palmer's recent paper [3].

The general cases of real tones (sinusoidal signals) and of many tones are, in a sense, extensions of the case studied here. The presence of several cysoidal signals introduces complexity in the bounds, ML estimation, and practical algorithms. These matters have been studied but are not reported here [2].

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